

Energy flux distribution in a two-temperature Ising model

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Abstract. The nonequilibrium steady state of an infinite-range Ising model is studied. The steady state is obtained by dividing the spins into two groups and attaching them to two heat baths generating spin flips at different temperatures. In the thermodynamic limit, the resulting dynamics can be solved exactly, and the probability flow in the phase space can be visualized. We can calculate the steady state fluctuations far from equilibrium and, in particular, we find the exact probability distribution of the energy current in both the high- and low-temperature phase.

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1. Introduction

The infinite range Ising model, in which each individual spin interacts with the remaining $N - 1$ ones has served as a useful testbench for many ideas in various subfields of statistical physics, ranging from critical dynamics to spin glasses. The reason is twofold: it is relatively easy to come up with exact yet nontrivial results for this system (in the large system limit at least), while at the same time it stands for a genuine interacting system which further possesses a phase transition from a disordered into an ordered state as the temperature is lowered. In the present work we will follow Ruijgrok and Tjon [1] who, by endowing the system with spin-flip dynamics, provided the first example of an exactly solvable critical dynamics problem back in 1973. In the meantime the subjects of interest have drifted towards other issues, but the technical motivations for using the infinite range Ising model remain. Our physical motivations have their roots in the study and identification of the generic properties of nonequilibrium steady-states (NESS).

Among the very few exact statements that can be made on NESS, the recent fluctuation –or Gallavotti–Cohen– theorem [2] plays a prominent role. This theorem is a symmetry property of the entropy current distribution function. As such it provides, at least formally, a prescription for obtaining an infinite set of Green–Kubo like relations connecting current fluctuations to the system’s response upon external forcing. While establishing a Gallavotti–Cohen theorem for Markov processes can be achieved in general terms [3, 4], explicit computations of current distribution functions are rare. For systems *close* to equilibrium such as boundary-driven lattice gases, Bodineau and Derrida [5], and Bertini *et al.* [6] have provided a general method for computing current distribution functions from the sole knowledge of the Onsager response coefficients. However for the generic case of systems maintained in a steady-state *far* from equilibrium, no general principle has hitherto been unraveled. To the best of our knowledge a single exact calculation exists for the totally asymmetric exclusion process, a degenerate case for which the Gallavotti–Cohen theorem (in the version of Lebowitz and Spohn [4]) does not hold. For this reason we have sought to exhibit a system driven far from equilibrium for which –albeit mean-field– such a current distribution function is accessible: an Ising model in which spins are connected to heat baths at different temperatures.

There are several ways of driving the infinite range Ising model into a NESS. One that is inspired from a series of recent works [7, 8, 9, 10] consists in coupling spins to independent heat baths, thus creating a macroscopic energy current by means of a bulk drive. In the particular version we have coined, N Ising spins σ_i have a ferromagnetic interaction energy

$$\mathcal{H}[\{\sigma_i\}] = -\frac{1}{2N} \left(\sum_i \sigma_i \right)^2 \quad (1)$$

and the dynamics of $N/2$ spins is generated by a heat bath of inverse temperature β_1 while the dynamics of the remaining half of the spins is driven by another heat bath at a different inverse temperature β_2 . This state of affairs, namely the existence of two heat baths at unequal temperatures, leads to a steady energy current flowing through the spins from the warmer bath towards the colder one.

The physical results we have obtained are concerned with the steady-state measure and the energy current distribution function. We have been able to provide an exact solution to the Fokker-Planck equation governing the probability distribution of magnetization fluctuations, thus leading to the first example of an N -body nonequilibrium system for which probability flow lines in phase space can actually be visualized. Our second achievement is to have provided the large deviation function of the energy current. The methods we have resorted to rest on the various formulations of the master equation governing the microscopic dynamics. On one hand, by means of a Van Kampen [11] expansion of the magnetizations around their mean values, we have obtained solvable Fokker-Planck equation describing steady-state fluctuations. Technically, this amounts to finding an eigenfunction of the master equation evolution operator. On the other hand, by means of a mapping of the master equation onto a quantum hamiltonian, we have determined the energy current distribution function, which, in technical terms, has proved to be an eigenvalue problem.

Our first task will be to provide an accurate description of the stationary state distribution (Sec. 3), which we will present after having precisely characterized our model and its steady-state properties (Sec. 2). Sec. 4 will be devoted to a complete study of the energy current distribution function, in the light of the Gallavotti–Cohen theorem. Our concluding remarks will be followed by two appendices generalizing the aforementioned results to an infinite-range Ising model in contact with more than two heat baths.

2. A two temperature Ising model: phase diagram and steady-state properties

2.1. Microscopic dynamics

The energy of a configuration of N Ising spins $\boldsymbol{\sigma} = \{\sigma_i\}$ is given by

$$\mathcal{H}[\boldsymbol{\sigma}] = -\frac{M^2}{2N}, \quad M = \sum_{i=1}^N \sigma_i \quad (2)$$

We now divide the N spins into two groups with labels 1 and 2 of $N/2$ spins each. A spin σ_j from set 1 flips with a rate

$$\forall j \in 1, \quad w_1(\sigma_j \rightarrow -\sigma_j) = e^{-\beta_1 \sigma_j M/N} \quad (3)$$

Spins from group 1 try to equilibrate at inverse temperature β_1 with respect to \mathcal{H} . Similarly, a spin from group 2 flips according to

$$\forall j \in 2, w_2(\sigma_j \rightarrow -\sigma_j) = e^{-\beta_2 \sigma_j M/N} \quad (4)$$

This is the infinite-range counterpart to the one-dimensional systems considered by Rácz and Zia [7], and Schmittmann and Schmuser [8]. Denoting by

$$\beta = \frac{\beta_1 + \beta_2}{2}, \quad \varepsilon = \frac{\beta_1 - \beta_2}{2} \quad (5)$$

we see that when the temperatures are equal, $\beta_1 = \beta_2 = \beta$, or $\varepsilon = 0$, the system reaches equilibrium at temperature β . This is because the rates (3,4) then satisfy detailed balance with respect to the Gibbs distribution $Z^{-1}e^{-\beta\mathcal{H}}$. Though the precise expressions of the rates we have chosen differ from the original Glauber rates, they possess the same qualitative properties with some advantages in the large-system limit discovered by Ruijgrok and Tjon [1].

2.2. Phase diagram

Introducing the mean magnetizations

$$m_1 = \frac{1}{N} \langle \sum_{j \in 1} \sigma_j \rangle, \quad m_2 = \frac{1}{N} \langle \sum_{j \in 2} \sigma_j \rangle, \quad m = m_1 + m_2 \quad (6)$$

we may find the following evolution equations for the averages:

$$\frac{dm_1}{dt} = -2m_1 \cosh \beta_1 m + \sinh \beta_1 m, \quad \frac{dm_2}{dt} = -2m_2 \cosh \beta_2 m + \sinh \beta_2 m \quad (7)$$

from which one deduces that in the steady-state (provided it exists)

$$m = \frac{1}{2} (\tanh \beta_1 m + \tanh \beta_2 m) \quad (8)$$

Interestingly, although the transition rates (3,4) are different from the standard Glauber rates, they lead to the same steady-state average magnetization. From (8) we deduce that in the steady-state the system undergoes a second order phase transition from a high-temperature disordered state at $\beta < 1$ in which $m_1 = m_2 = m = 0$ to a low-temperature ordered (doubly degenerate) state at $\beta > 1$ with nonzero magnetizations. In the $\beta \rightarrow 1^+$ limit at ε fixed one finds

$$m \simeq \pm \frac{\sqrt{3}}{\sqrt{1+3\varepsilon^2}} \sqrt{\beta-1}, \quad m_1 \simeq \pm \frac{1+\varepsilon}{2} \frac{\sqrt{3}}{\sqrt{1+3\varepsilon^2}} \sqrt{\beta-1}, \quad m_2 \simeq \pm \frac{1-\varepsilon}{2} \frac{\sqrt{3}}{\sqrt{1+3\varepsilon^2}} \sqrt{\beta-1} \quad (9)$$

According to the magnitude of the nonequilibrium drive ε it may be seen that the ordered state may be either ferromagnetic ($|\varepsilon| < 1$) or antiferromagnetic ($|\varepsilon| > 1$, if one allows for negative temperatures, as we shall discuss in our conclusion in Sec. 5).

2.3. Entropy and energy currents

Following the prescription of Lebowitz and Spohn [4] we may define a time integrated instantaneous entropy current by

$$\mathcal{Q}_S(t) = \ln \frac{W(\boldsymbol{\sigma}^{(0)} \rightarrow \boldsymbol{\sigma}^{(1)})}{W(\boldsymbol{\sigma}^{(1)} \rightarrow \boldsymbol{\sigma}^{(0)})} \cdots \frac{W(\boldsymbol{\sigma}^{(k-1)} \rightarrow \boldsymbol{\sigma}^{(k)})}{W(\boldsymbol{\sigma}^{(k)} \rightarrow \boldsymbol{\sigma}^{(k-1)})} \quad (10)$$

where $\boldsymbol{\sigma}^{(0)} = \boldsymbol{\sigma}(0), \dots, \boldsymbol{\sigma}^{(k)} = \boldsymbol{\sigma}(t)$ is the sequence of states occupied by the system over the time interval $[0, t]$ (this is the history of the system between 0 and t). The rates $W(\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma}')$ of hopping from configuration $\boldsymbol{\sigma}$ to configuration $\boldsymbol{\sigma}'$ between t and $t + dt$ are easily deduced from (3,4). Inserting the explicit expressions for the $W(\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma}')$ leads to

$$\mathcal{Q}_S(t) = -\beta (\mathcal{H}[\boldsymbol{\sigma}(t)] - \mathcal{H}[\boldsymbol{\sigma}(0)]) + \varepsilon Q(t) \quad (11)$$

where we identify Q as the integrated energy current:

$$Q(t) = -\frac{2}{N} \sum_{n=0}^k (\pm) \sigma_{j_n} (M_n - \sigma_{j_n}) \quad (12)$$

where σ_{j_n} is the spin being flipped at time n and M_n is the total magnetization at that moment. The sign $+$ (resp. $-$) corresponds to flipping a spin from group 1 (resp. 2). Note that $\mathcal{H}[\boldsymbol{\sigma}(t)] - \mathcal{H}[\boldsymbol{\sigma}(0)]$ being bounded over time, $\mathcal{Q}_S(t)$ and $\varepsilon Q(t)$ have the same large deviation functions. It is clear that on average,

$$J_\varepsilon = \frac{\langle Q(t) \rangle}{t} = -\frac{2}{N} \left\langle \sum_{j \in 1} \sigma_j (M - \sigma_j) e^{-\beta_1 \sigma_j M/N} - \sum_{j \in 2} \sigma_j (M - \sigma_j) e^{-\beta_2 \sigma_j M/N} \right\rangle \quad (13)$$

While interpreting $\mathcal{Q}_S(t)$ as an integrated entropy current requires an elaborate reasoning [4], the physical meaning of J_ε as an energy current is much more intuitive. Indeed, the total energy of the system is constant on average in the steady-state:

$$\frac{d\langle \mathcal{H} \rangle}{dt} = 0 = -(J_1 + J_2) \quad (14)$$

where J_α is the energy flux due to spin-flips caused by heat-bath α (for instance J_1 is the first term on the rhs of (13)). The quantity $J_\varepsilon = J_1 - J_2$ is therefore a measure of the energy flowing from group 1 towards group 2. Hence the related entropy current J_S must read

$$J_S = \beta_1 J_1 + \beta_2 J_2 = \varepsilon J_\varepsilon \quad (15)$$

This interpretation of $\varepsilon J_\varepsilon$ as an entropy current has been discussed, on the grounds of phenomenological thermodynamics, by R acz and Zia [7].

As described in appendix B, there is no immediate link between the entropy current and an energy current for a system in contact with more than two heat baths.

3. Stationary state distribution

3.1. Van Kampen expansion and Fokker-Planck equation

In this section we derive a Fokker-Planck equation governing the probability $P(x_1, x_2, t)$ of observing the following fluctuations of the spin magnetizations:

$$x_\alpha = \frac{\sum_{j \in \alpha} \sigma_j - N m_\alpha}{\sqrt{N}}, \quad \alpha = 1, 2 \quad (16)$$

This is the Van Kampen [11] expansion of the master equation around the mean magnetizations m_α . The \sqrt{N} rescaling is precisely designed for the x_α to have order 1 fluctuations. We find that $P(x_1, x_2, t)$ satisfies the following Fokker-Planck equation:

$$\partial_t P = -\partial_{x_1} \mathcal{J}_1 - \partial_{x_2} \mathcal{J}_2 \quad (17)$$

where the probability current is given by

$$\mathcal{J}_\alpha = f_\alpha(x_1, x_2)P - D_\alpha \partial_{x_\alpha} P \quad (18)$$

The two-dimensional force (f_1, f_2) does not derive from a potential unless both heat baths are at the same temperature. The general expression of the force components is

$$\begin{aligned} f_1(x_1, x_2) &= ((\beta_1 - 2)x_1 + \beta_1 x_2) \cosh \beta_1 m - 2\beta_1(x_1 + x_2)m_1 \sinh \beta_1 m \\ f_2(x_1, x_2) &= ((\beta_2 - 2)x_2 + \beta_2 x_1) \cosh \beta_2 m - 2\beta_2(x_1 + x_2)m_2 \sinh \beta_2 m \end{aligned} \quad (19)$$

The diffusion constants are given by

$$D_\alpha = \cosh \beta_\alpha m - 2m_\alpha \sinh \beta_\alpha m = \sqrt{1 - 4m_\alpha^2} \quad (20)$$

In the high temperature phase, using that $\beta_{1/2} = \beta \pm \varepsilon$, this may easily be cast in the following form

$$f_1(x_1, x_2) = -\partial_{x_1} U_\varepsilon + \varepsilon x_2, \quad f_2(x_1, x_2) = -\partial_{x_2} U_\varepsilon - \varepsilon x_1 \quad (21)$$

where the potential energy has the expression

$$U_\varepsilon(x_1, x_2) = (1 - \beta) \frac{(x_1 + x_2)^2}{2} + \frac{(x_1 - x_2)^2}{2} - \varepsilon \frac{x_1^2 - x_2^2}{2} \quad (22)$$

In the high temperature phase, to which the ensuing analysis will be confined for simplicity (a general solution is provided in appendix A), where $m_1 = m_2 = 0$, we thus have to solve

$$\partial_t P = 0 = -\partial_{x_1} \mathcal{J}_1 - \partial_{x_2} \mathcal{J}_2 \quad (23)$$

where the probability current reduces to

$$\mathcal{J}_1 = ((\beta - 2)x_1 + \beta x_2 + \varepsilon(x_1 + x_2))P - \partial_{x_1} P, \quad \mathcal{J}_2 = ((\beta - 2)x_2 + \beta x_1 - \varepsilon(x_1 + x_2))P - \partial_{x_2} P \quad (24)$$

When $\beta_1 = \beta_2 = \beta$, that is in equilibrium, the distribution reads

$$P(x_1, x_2) \sim \exp \left[-(1 - \beta) \frac{(x_1 + x_2)^2}{2} - \frac{(x_1 - x_2)^2}{2} \right] \quad (25)$$

We may find the exact solution to the Fokker-Planck equation by having the intuition, following [11], that the effective potential U_{eff} defined by

$$P(x_1, x_2) = Z^{-1} \exp(-U_{\text{eff}}) \quad (26)$$

will be quadratic in terms of x_1 and x_2 . This is suggested by the force being linear in x_1 and x_2 . And indeed this naive assumption leads to the effective potential U_{eff} given by

$$U_{\text{eff}}(x_1, x_2) = \frac{1-\beta}{2}(x_1 + x_2)^2 + \frac{2}{4+J^2} (x_1[1-J/2] - x_2[1+J/2])^2 \quad (27)$$

where we have introduced the constant $J \equiv \frac{2\varepsilon}{2-\beta}$. It is then an easy task to compute the mean energy current J_ε :

$$\begin{aligned} J_\varepsilon &= \left\langle \sum_{j \in 1} (-2\sigma_j^z (M^z - \sigma_j^z)/N) e^{-\beta_1 \sigma_j^z M^z / N} - \sum_{j \in 2} (-2\sigma_j^z (M^z - \sigma_j^z)/N) e^{-\beta_2 \sigma_j^z M^z / N} \right\rangle \\ &= 2\varepsilon \langle (x_1 + x_2)^2 \rangle - 2 \langle (x_1^2 - x_2^2) \rangle \\ &= \frac{2\varepsilon}{2-\beta} = J \end{aligned} \quad (28)$$

In terms of the magnetization fluctuations the solution to the Fokker-Planck equation reads

$$P(x_1, x_2) \sim \exp \left[-\frac{1-\beta}{2}(x_1 + x_2)^2 - \frac{2}{4+J^2} (x_1[1-J/2] - x_2[1+J/2])^2 \right] \quad (29)$$

A few comments are in order. In spite of the phase space being only two-dimensional, this is just enough to allow for inhomogeneous currents to flow (contrary to a one-dimensional phase space). While the total magnetization has global fluctuations equal to those of a system in equilibrium at β , it may be seen that the magnetization difference between the two spin groups is increased with respect to its equilibrium counterpart in the presence of a current:

$$\langle (x_1 - x_2)^2 \rangle_{J \neq 0} - \langle (x_1 - x_2)^2 \rangle_{\text{eq}, J=0} = \frac{J^2}{4} \frac{2-\beta}{1-\beta} \quad (30)$$

This provides an example of a nonequilibrium drive giving rise to an *increase* of fluctuations, rather than to a *decrease* (as is usually noted, *e.g.* in driven lattice gases [12, 13] and spin chains [14]). We now turn to an analysis of the probability flow lines.

3.2. Flow lines

In equilibrium, by definition, there is no probability current, while in a NESS there are steady (probability) currents. The flow lines, namely the set of points (x_1, x_2) such that $J_1(x_1, x_2)dx_2 - J_2(x_1, x_2)dx_1 = 0$, in phase space turn out to be ellipses, as shown in Fig (1). For $J \neq 0$ the flow lines are ellipses of equation

$$(1-\beta) \left(1 + \frac{J^2}{4} \right) (x_1 + x_2)^2 + (x_1[1-J/2] - x_2[1+J/2])^2 = C^2 \quad (31)$$

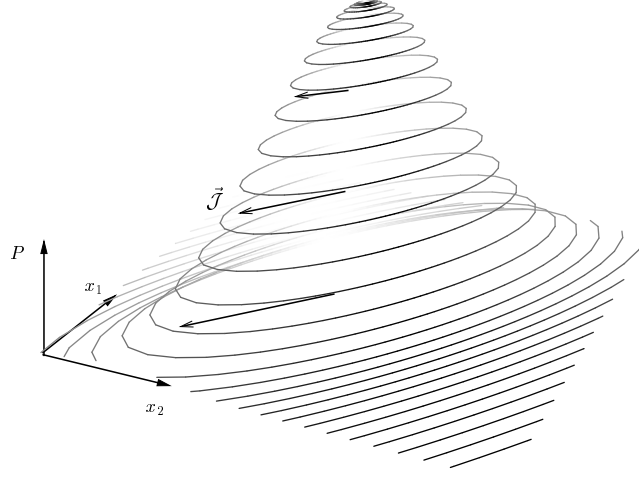


Figure 1. The (x_1, x_2) axes denote the deviations from the group 1 and 2 magnetizations. Flow lines of the probability current \vec{J} are represented for different values of the constant C in (31). The vertical axis is the probability $P(x_1, x_2)$, illustrating that the flow lines coincide with the isoprobability contours. On this figure, $\beta = 0.8$ and $\varepsilon = 0.7$.

and they coincide with the isoprobability contours, a generic property of linear force driven systems. For an equilibrium system the flow lines can be seen to collapse onto a single point. Interestingly, the shape of the flow lines can be used to infer properties of the steady-state distribution: the departure from the ellipses will indicate deviations from linear forces in the Fokker–Planck equation.

3.3. Master equation and effective free energy

We characterize a *state* of our system by the “local” magnetizations $M_\alpha = \sum_{j \in \alpha} \sigma_j$ and we denote (in this paragraph only) by $m_\alpha = M_\alpha/N$. The steady-state solution to the master equation being denoted by $P_{\text{st}}(M_1, M_2)$, we define the effective free energy $f(m_1, m_2)$ by

$$f(m_1, m_2) = - \lim_{N \rightarrow \infty} \frac{\ln P_{\text{st}}(Nm_1, Nm_2)}{N} \quad (32)$$

We split f into an entropic contribution $s(m_1, m_2)$ and an effective energy $e(m_1, m_2)$. Whether in equilibrium or in a NESS, the entropic part is defined by the combinatoric factor for the number of configurations with M_1 and M_2 :

$$\begin{aligned} s(m_1, m_2) &= \frac{1}{N} \ln \left(\binom{\frac{N}{2}}{\frac{N+2M_1}{4}} \right) \left(\binom{\frac{N}{2}}{\frac{N+2M_2}{4}} \right) \\ &= - \sum_{\alpha} \left[\frac{1+2m_{\alpha}}{4} \ln \frac{1+2m_{\alpha}}{4} + \frac{1-2m_{\alpha}}{4} \ln \frac{1-2m_{\alpha}}{4} \right] \end{aligned} \quad (33)$$

This fully defines $e = f + s$. In equilibrium we have that $e_{\text{eq}}(m_1, m_2) = -\frac{\beta}{2}(m_1 + m_2)^2$, and we wish to find how the nonequilibrium drive modifies this result, namely what kind of effective interactions between the two groups of spins it generates. Since $\sim e^{N(s-e)}$ is a stationary solution to the master equation governed by the rates (3,4), we find that $e(m_1, m_2)$ is a solution to

$$\begin{aligned} 0 = & (1 - 2m_1) (e^{-\beta_1(m_1+m_2)-2\partial_1 e} - e^{+\beta_1(m_1+m_2)}) \\ & + (1 + 2m_1) (e^{+\beta_1(m_1+m_2)+2\partial_1 e} - e^{-\beta_1(m_1+m_2)}) \\ & + (1 - 2m_2) (e^{-\beta_2(m_1+m_2)-2\partial_2 e} - e^{+\beta_2(m_1+m_2)}) \\ & + (1 + 2m_2) (e^{+\beta_2(m_1+m_2)+2\partial_2 e} - e^{-\beta_2(m_1+m_2)}) \end{aligned} \quad (34)$$

To first order in ε one may verify that

$$e(m_1, m_2) = -\frac{1}{2}\beta m^2 - \frac{\varepsilon}{2-\beta}(m_1 - m_2)h(m) + \mathcal{O}(\varepsilon^2), \quad m = m_1 + m_2 \quad (35)$$

where the function h is the solution to the following first order ordinary differential equation:

$$(m - \tanh \beta m) \frac{dh}{dm} + h(m) - (2 - \beta)m = 0, \quad h(0) = 0 \quad (36)$$

For instance, as $m \rightarrow 0$,

$$h(m) = m - \frac{\beta^3}{3(1 + 3(1 - \beta))}m^3 + \mathcal{O}(m^5) \quad (37)$$

thus recovering the leading term of the high temperature Van Kampen expansion. At this stage we have completed our description of the steady-state properties. Note that the structure of (35) has flavors of the much more complex one found by Derrida *et al.* [15] in the framework of the asymmetric exclusion process for the effective free energy of a given density profile.

4. Energy current distribution

This section is devoted to determining the large deviation function of the time integrated energy current.

4.1. Modified master equation

Let $Q(t)$ be the fluctuating energy current integrated over the time interval $[0, t]$ as defined in (12). We are interested in $p(Q, t)$, the probability that $Q(t) = Q$ at time t or in its generating function $\hat{p}(\lambda, t) = \langle e^{-\lambda Q} \rangle$. It is possible to write a master equation for

$$P(M_1, M_2, Q, t) = \text{Prob}\left\{\sum_{j \in 1} \sigma_j = M_1 \text{ and } \sum_{j \in 2} \sigma_j = M_2 \text{ and } Q(t) = Q\right\} \quad (38)$$

Inserting the explicit expressions of the transition rates we arrive at

$$\partial_t P = \frac{N/2 + M_1 + 2}{N} e^{-\beta_1 \frac{M+2}{N}} P(M_1 + 2, M_2, Q - 2\frac{M+1}{N}, t) - \frac{N/2 - M_1}{N} e^{+\beta_1 \frac{M}{N}} P(M_1, M_2, Q, t)$$

$$\begin{aligned}
& + \frac{N/2 - M_1 + 2}{N} e^{+\beta_1 \frac{M-2}{N}} P(M_1 - 2, M_2, Q + 2 \frac{M-1}{N}, t) - \frac{N/2 + M_1}{N} e^{-\beta_1 \frac{M}{N}} P(M_1, M_2, Q, t) \\
& + \frac{N/2 + M_2 + 2}{N} e^{-\beta_2 \frac{M+2}{N}} P(M_1, M_2 + 2, Q - 2 \frac{M+1}{N}, t) - \frac{N/2 - M_2}{N} e^{+\beta_2 \frac{M}{N}} P(M_1, M_2, Q, t) \\
& + \frac{N/2 - M_2 + 2}{N} e^{+\beta_2 \frac{M-2}{N}} P(M_1, M_2 - 2, Q + 2 \frac{M-1}{N}, t) - \frac{N/2 + M_2}{N} e^{-\beta_2 \frac{M}{N}} P(M_1, M_2, Q, t)
\end{aligned} \tag{39}$$

Going to the generating function

$$\hat{P}(M_1, M_2, \lambda, t) = \sum_Q e^{-\lambda Q} P(M_1, M_2, Q, t) \tag{40}$$

and setting $|\Psi(\lambda, t)\rangle = \sum_{M_1, M_2} \hat{P}(M_1, M_2, \lambda, t) |M_1, M_2\rangle$, we may rewrite Eq. (39) in the following form

$$\frac{d|\Psi(\lambda, t)\rangle}{dt} = -\hat{H}(\lambda) |\Psi(\lambda, t)\rangle \tag{41}$$

where the operator $\hat{H}(\lambda)$ reads

$$\hat{H}(\lambda) = \sum_{j \in 1} \left(1 - \sigma_j^x e^{+2\lambda \sigma_j^z (M^z - \sigma_j^z)/N} \right) e^{-\beta_1 \sigma_j^z M^z / N} + \sum_{j \in 2} \left(1 - \sigma_j^x e^{-2\lambda \sigma_j^z (M^z - \sigma_j^z)/N} \right) e^{-\beta_2 \sigma_j^z M^z / N} \tag{42}$$

The asymptotic behavior of

$$\hat{p}(\lambda, t) = \sum_{M_1, M_2} \hat{P}(M_1, M_2, \lambda, t) = \langle \mathbf{p} | e^{-\hat{H}(\lambda)t} | \Psi(0) \rangle, \quad \langle \mathbf{p} | = \sum_{M_1, M_2} \langle M_1, M_2 | = \text{projection state} \tag{43}$$

will be governed by the largest eigenvalue $\mu(\lambda)$ of $-\hat{H}(\lambda)$ in the sense that $\lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{p}(\lambda, t) = \mu(\lambda)$, which we now set out to determine. Before embarking into technicalities it is convenient, but by no means compulsory, to perform a similitude transformation on $\hat{H}(\lambda)$

$$\begin{aligned}
\hat{H}_s(\lambda) &= e^{-\beta(M^z)^2/4N} \hat{H}(\lambda) e^{+\beta(M^z)^2/4N} \\
&= \sum_{j \in 1} \left(e^{-\beta_1 \sigma_j^z M^z / N} - \sigma_j^x e^{+(2\lambda - \varepsilon) \sigma_j^z M^z / N - (2\lambda + \beta)/N} \right) \\
&\quad + \sum_{j \in 2} \left(e^{-\beta_2 \sigma_j^z M^z / N} - \sigma_j^x e^{-(2\lambda - \varepsilon) \sigma_j^z M^z / N + (2\lambda - \beta)/N} \right)
\end{aligned} \tag{44}$$

The transformation (44) does not have the same effect as that conducted by Ruijgrok and Tjon [1] –it does not make the resulting operator Hermitian– but it serves the same practical purpose: calculations are performed in a more convenient way where the system symmetries (upon exchanging the roles of 1 and 2) are made obvious. In terms of its symmetrized counterpart $\hat{H}_s(\lambda)$, we have that

$$\left(\hat{H}_s(\lambda) \right)^\dagger = \hat{H}_s(\varepsilon - \lambda^*) \tag{45}$$

An important consequence of symmetry (45) is that, for λ real, both $\hat{H}_s(\lambda)$ and $\hat{H}_s(\varepsilon - \lambda)$ have the same spectrum, hence

$$\mu(\lambda) = \mu(\varepsilon - \lambda) \tag{46}$$

This is the Gallavotti–Cohen theorem. A direct consequence for the energy current large deviation function $\pi(q) = \frac{1}{t} \ln p(Q = qt, t)$ is that

$$\pi(q) - \pi(-q) = \varepsilon q \quad (47)$$

where we have used that $\pi(q) = \max_{\lambda} \{\mu(\lambda) + \lambda q\}$. Another useful consequence of (45) is that for $\lambda \in \frac{\varepsilon}{2} + i\mathbb{R}$, $\hat{H}_s(\lambda)$ is Hermitian, which will justify diagonalization in that region of the λ complex plane.

4.2. Mapping to a free boson problem

We introduce, following Ruijgrok and Tjon [1], bosonic operators $a_{\alpha}, a_{\alpha}^{\dagger}$ ($\alpha = 1, 2$) to describe magnetizations 1 and 2 in the vicinity of the paramagnetic state:

$$M_{\alpha}^x = N/2 - 2a_{\alpha}^{\dagger}a_{\alpha}, \quad M_{\alpha}^y = -i\sqrt{N/2}(a_{\alpha}^{\dagger} - a_{\alpha}), \quad M_{\alpha}^z = \sqrt{N/2}(a_{\alpha}^{\dagger} + a_{\alpha}) \quad (48)$$

The relations (48) hold provided we are interested in states such that the number operator $a_{\alpha}^{\dagger}a_{\alpha}$ remain of order unity, that is much smaller than \sqrt{N} . In terms of these operators we find that

$$\hat{H}_s(\lambda) = \frac{1}{2} \begin{pmatrix} a_1^{\dagger} & a_1 & a_2^{\dagger} & a_2 \end{pmatrix} \Gamma(\lambda) \begin{pmatrix} a_1^{\dagger} \\ a_1 \\ a_2^{\dagger} \\ a_2 \end{pmatrix} + \frac{1}{2}(\beta^2 + 4\lambda(\varepsilon - \lambda)) \quad (49)$$

with

$$\Gamma = \begin{pmatrix} Z - 2\lambda & Z + 2 & Z & Z - (2\lambda - \varepsilon) \\ Z + 2 & Z + 2(\lambda - \varepsilon) & Z + 2\lambda - \varepsilon & Z \\ Z & Z + 2\lambda - \varepsilon & Z + 2\lambda & Z + 2 \\ Z - (2\lambda - \varepsilon) & Z & Z + 2 & Z - 2(\lambda - \varepsilon) \end{pmatrix} \quad (50)$$

where $Z = -\frac{1}{2}\beta(2 - \beta) + 2\lambda(\varepsilon - \lambda)$. For normalization purposes [16] it is necessary to define an auxiliary matrix $\tilde{\Gamma}$ built from Γ by equating to zero in the latter all matrix elements connecting two creation or two annihilation operators. It is then a simple matter [16] to determine not only the ground-state but also the spectrum of $\hat{H}_s(\lambda)$. To do so we introduce the matrix Ω defined as

$$\Omega = \begin{pmatrix} 0 & i\omega & 0 & 0 \\ -i\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & i\omega \\ 0 & 0 & -i\omega & 0 \end{pmatrix} \quad (51)$$

We must now evaluate the quantity

$$\mu(\lambda) = \frac{1}{2} \int \frac{d\omega}{2\pi} \ln \frac{\det(\tilde{\Gamma} + \Omega)}{\det(\Gamma + \Omega)} - \frac{1}{2}(\beta^2 + 4\lambda(\varepsilon - \lambda)) \quad (52)$$

Given that

$$\det(\Gamma + \Omega) = (\omega^2 + \omega_+^2)(\omega^2 + \omega_-^2) \quad (53)$$

with

$$\omega_{\pm}(\lambda) = \sqrt{(2 - \beta)^2 + 4\lambda(\varepsilon - \lambda)} \pm \sqrt{\beta^2 + 4\lambda(\varepsilon - \lambda)} \quad (54)$$

and that similarly

$$\det(\tilde{\Gamma} + \Omega) = (\omega^2 + \tilde{\omega}_+^2)(\omega^2 + \tilde{\omega}_-^2) \quad (55)$$

with

$$\tilde{\omega}_{\pm}(\lambda) = \frac{1}{2} \left((2 - \beta)^2 + 4\lambda(\varepsilon - \lambda) \pm \sqrt{[(2 - \beta)^2 + 4\lambda(\varepsilon - \lambda)] [\beta^2 + 4\lambda(\varepsilon - \lambda)]} \right) \quad (56)$$

we arrive at the following result:

$$\mu(\lambda) = \frac{1}{2}(\tilde{\omega}_+ + \tilde{\omega}_- - \omega_+ - \omega_-) - \frac{1}{2}(\beta^2 + 4\lambda(\varepsilon - \lambda)) \quad (57)$$

which simplifies into

$$\mu(\lambda) = 2 - \beta - \sqrt{(2 - \beta)^2 + 4\lambda(\varepsilon - \lambda)} \quad (58)$$

By the same token we obtain the spectrum of $\hat{H}(\lambda)$, whose eigenvalues are given by

$$\text{Sp}(\hat{H}(\lambda)) = \{\omega_+(\lambda)\ell + \omega_-(\lambda)\ell' - \mu(\lambda)\}_{\ell, \ell' \in \mathbb{N}} \quad (59)$$

Specializing to $\lambda = 0$ we obtain as a side result the spectrum of the master equation evolution operator, whose slowest relaxation time is given by $\omega_+^{-1}(0) = (2(1 - \beta))^{-1}$. This again matches the results of [1].

Given that $\pi(q)$ and $\mu(\lambda)$ are the Legendre transforms of each other we arrive at the explicit form of the current large deviation function $\pi(q)$

$$\pi(q) = \frac{\varepsilon}{2}q + 2 - \beta - \sqrt{(2 - \beta)^2 + \varepsilon^2} \sqrt{4 + q^2} \quad (60)$$

and it has the graph shown in Fig. 2.

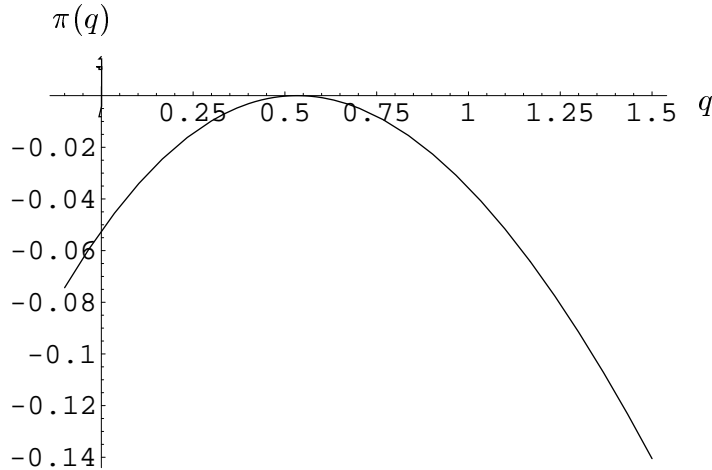


Figure 2. This is the plot of the energy flux large deviation function $\pi(q)$ (vertical axis) as a function of q (horizontal axis) at $\beta = 0.5$ and $\varepsilon = 0.4$.

4.3. Energy current in the low temperature phase

Similar methods allowed us to express the generating function of the cumulants of $Q(t)$ in the low temperature phase, at $\beta \geq 1$. The final result is

$$\mu(\lambda) = c_1 + c_2 - \frac{1}{2} \left(\frac{\beta_1}{c_1} + \frac{\beta_2}{c_2} \right) - \sqrt{\left[c_1 + c_2 - \frac{1}{2} \left(\frac{\beta_1}{c_1} + \frac{\beta_2}{c_2} \right) \right]^2 + \frac{4}{c_1 c_2} \lambda (\varepsilon - \lambda)} \quad (61)$$

where $c_\alpha = \cosh \beta_\alpha(m_1 + m_2) = 1/\sqrt{1 - 4m_\alpha^2}$ and where m_α is the stationary solution of (7), and this is a function of β and ε . However the current is not the order parameter of the phase transition, therefore nothing dramatic is expected to occur for $\mu(\lambda)$ at $\beta = 1$. An important difference with the high temperature result must be emphasized: in the low temperature ordered regime the current is a nonlinear function of ε , as plotted in Fig. 3. The similar mathematical structure of $\mu(\lambda)$ in the high and low temperature

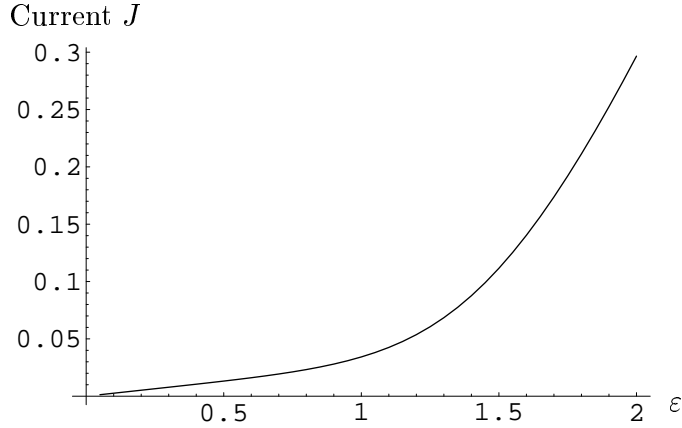


Figure 3. Plot of the average energy current J as a function of $\varepsilon \in [0, \beta]$ at $\beta = 2$ in the ordered phase.

phases seems to be generically related to Langevin equations with linear forces [17].

The energy current at fixed drive $\varepsilon = 0.5$ as a function of $\beta \in [0.5, 2]$ represented in Fig. 4 shows that, from the disordered to the ordered phase, the current remains finite and continuous, though it develops a cusp at the critical point $\beta = 1$.

4.4. Green-Kubo relations

Exploiting the explicit formula (58) for $\mu(\lambda)$ we find, after differentiation with respect to λ once and twice, that

$$\frac{\langle Q \rangle}{t} = J = \frac{2\varepsilon}{2 - \beta}, \quad \frac{\langle Q^2 \rangle_c}{t} = \frac{4}{2 - \beta} + \frac{4}{(2 - \beta)^3} \varepsilon^2 \quad (62)$$

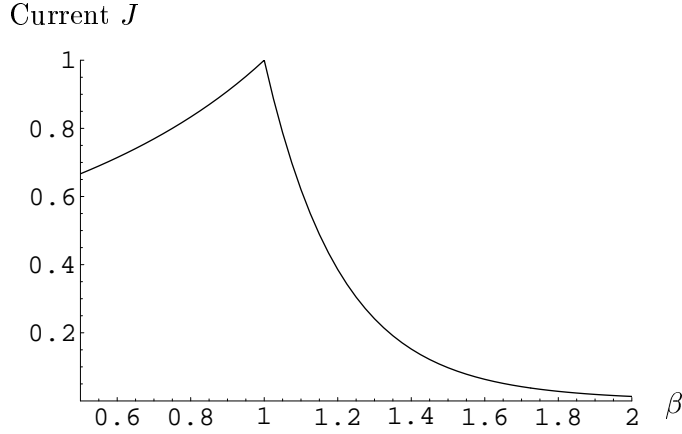


Figure 4. Plot of the average energy current J as a function of $\beta \in [0.5, 2]$ at $\varepsilon = 0.5$.

Note that defining the diffusion coefficient $D(\beta)$ as the response to an external drive and $\sigma(\beta)$ as the variance of the current fluctuations we find

$$D(\beta) = \left. \frac{\partial J(\beta_1, \beta_2)}{\partial \beta_1} \right|_{\beta_1=\beta_2=\beta} = \frac{1}{2-\beta}, \quad \sigma(\beta) = \left. \frac{\langle Q^2 \rangle_c}{t} \right|_{\beta_1=\beta_2=\beta} \quad (63)$$

With these expressions one may verify an integral formulation of the Green-Kubo relation

$$2 \int_{\beta_2}^{\beta_1} d\beta \frac{D(\beta)}{\sigma(\beta)} = \varepsilon \quad (64)$$

Nevertheless the sole knowledge of $D(\beta)$ and $\sigma(\beta)$ does not give access to the full distribution $\mu(\lambda)$, as opposed to the cases studied by Bodineau and Derrida [5] by means of an additivity principle or by Bertini *et al.* [18, 6] who resorted to fluctuating hydrodynamics [18, 6]. In order the latter approaches to hold, the typical current must scale to zero with the system size at fixed (intensive) external field. This the second example, aside from the extensively studied Asymmetric Exclusion Process [19], of an interacting system, albeit mean-field, in which the entropy (or energy) current can be computed exactly, with the additional property in our case that the Gallavotti–Cohen theorem is fulfilled, hence generalized Green–Kubo relations as well.

5. Final comments

We have been able to present explicit and exact results for the steady-state of a system made of interacting spins driven far from equilibrium by heat baths at different temperatures. The system described exhibits a ferromagnetic-to-paramagnetic phase transition. The simplicity of some of our results, like Gaussian fluctuations for the magnetizations, are admittedly an artifact of our infinite-range, mean-field, model. Nevertheless, due to easier mathematics, we have been able to precisely describe the probability flow lines, ellipses in a two-dimensional phase space. Other concepts arising within the framework

of dynamical systems theory, like that of topological pressure, once transposed to our model, might equally lend themselves to analytical approaches.

Our other result of interest concerns the computation of an energy current distribution for a system far from equilibrium, that cannot be described by fluctuating hydrodynamics, although it falls within the scope of the Gallavotti–Cohen theorem for Markov processes [4]. To our knowledge, this is the first one of this sort. It is only a first step towards the desirable, but remote, goal of characterizing stationary systems driven very far from equilibrium.

Among the prospects uncovered by the present work, we mention the extension of the urn model of Bena *et al.* [20] analyzed in the light of Greenblatt and Lebowitz’ comment [21]. In spite of being genuinely nonequilibrium, our model will most probably not display any surprises as far as the Yang–Lee zeros of the partition function are concerned, simply because the phase transition that takes place at $\beta = 1$ is akin to its equilibrium counterpart (and belongs to the same universality class). However the urn model may be very well defined for negative temperatures. Preliminary studies indicate that such rates open the door to limit cycles and chaotic behavior that we shall explore in future studies.

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Appendix A: Fokker–Planck equation for n heat baths

In the disordered phase, the Ising system is split in n parts of magnetization fluctuation x_α ($1 \leq \alpha \leq n$), each of them equilibrated with a bath at inverse temperature $\beta_\alpha = \beta + \varepsilon_\alpha$, where β is the average of the β_α ’s. The Fokker–Planck stationary equation for the whole system can be written $\partial_\alpha \mathcal{J}_\alpha = 0$ (we will use summation convention over repeated indices). The \mathcal{J}_α ’s are the components of the probability current, of expression :

$$\mathcal{J}_\alpha = -\partial_\alpha P + f_{\alpha\gamma} x_\gamma P \quad (65)$$

$$f_{\alpha\gamma} = -n\delta_{\alpha\gamma} + \beta_\alpha \quad (66)$$

This is a special case of a Fokker–Planck equation with linear forces $f_{\alpha\gamma} x_\gamma$. As f is a definite negative matrix, we know from general considerations [11] that its stationary state is Gaussian of covariance matrix :

$$\int_0^\infty dt e^{tf} e^{tf^T} \quad (67)$$

This matrix however proves difficult to be explicitated in the general case. Here we will restrict our analysis to the case where f can be diagonalized, as in our system, and use an indirect method to compute the result of (67). First note that if f is symmetric,

setting $U(x_1, \dots, x_n) = -\frac{1}{2}x^T f x$, we have $f_{\alpha\gamma}x_\gamma = -\partial_\alpha U$. In other words, the forces applied on the system are conservative and deriving from the potential U . In that case, we immediately find the stationary solution of the Fokker–Planck equation by requiring $\mathcal{J}_\alpha = 0$: this is, as expected, the Gibbs–Boltzman distribution $P(x) \sim e^{-U(x)}$, which corresponds to an equilibrium situation, and the defining absence of a probability current.

For f diagonalizable but not symmetric, no solution can be found imposing $\mathcal{J}_\alpha = 0$. Our aim is to parallel the resolution of the equilibrium case, by transforming the Fokker–Planck equation into some new equation $\partial_\alpha \mathcal{J}'_\alpha = 0$ whose solution can be found by requiring $\mathcal{J}'_\alpha = 0$. We first perform the change of variable $y = bx$, where b is a matrix such that $bfb^{-1} = \text{Diag}(\lambda_1, \dots, \lambda_n)$. The Fokker–Planck equation takes the form :

$$-b_{\alpha\alpha'}b_{\gamma\alpha'}\partial_\alpha\partial_\gamma P + \partial_\alpha(\lambda_\alpha y_\alpha P) = 0 \quad (68)$$

where now $\partial_\alpha = \partial/\partial y_\alpha$. The force term is symmetric, and can be formally written as deriving from some potential. There are many ways to split the first term of (68) so as to write it as a divergence. Let $A_{\alpha\gamma}$ be arbitrary constants. Writing, for $\alpha < \gamma$:

$$b_{\alpha\alpha'}b_{\gamma\alpha'}\partial_\alpha\partial_\gamma P = \partial_\alpha(A_{\alpha\gamma}b_{\alpha\alpha'}b_{\gamma\alpha'}\partial_\gamma P) + \partial_\gamma((1 - A_{\alpha\gamma})b_{\alpha\alpha'}b_{\gamma\alpha'}\partial_\alpha P) , \quad (69)$$

the FP equation now takes the announced form $\partial_\alpha \mathcal{J}'_\alpha = 0$, and requiring $\mathcal{J}'_\alpha = 0$ means (defining $\vec{b}_\alpha \cdot \vec{b}_\gamma = b_{\alpha\alpha'}b_{\gamma\alpha'}$) :

$$\underbrace{\begin{pmatrix} \vec{b}_1^2 & & & (2A_{\alpha\gamma}\vec{b}_\alpha \cdot \vec{b}_\gamma) \\ & \vec{b}_2^2 & & \\ & & \ddots & \\ (2(1 - A_{\gamma\alpha})\vec{b}_\alpha \cdot \vec{b}_\gamma) & & & \vec{b}_n^2 \end{pmatrix}}_L \vec{\nabla} P = \underbrace{\text{Diag}(\lambda_1, \dots, \lambda_n)}_\Lambda \vec{y} \quad (70)$$

This equation has a solution if, and only if $\Lambda^{-1}L$ is symmetric, which occurs when:

$$A_{\alpha\gamma} = \frac{\lambda_\gamma}{\lambda_\alpha + \lambda_\gamma} \quad (71)$$

In terms of the original x variables, we finally get the following expression :

$$P(x) = \exp\left(-\frac{1}{2}x^T b^T M_y^{-1} b x\right) \quad \text{with} \quad (M_y)_{\alpha\gamma} = \frac{2\vec{b}_\alpha \cdot \vec{b}_\gamma}{\lambda_\alpha + \lambda_\gamma} \quad (72)$$

in which we can read the result of (67) in the general case. When applying this result to the n baths Ising problem, we find :

$$P(x) = \exp\left(-\frac{1}{2}x^T M^{-1} x\right) \quad \text{with} : M_{\alpha\gamma} = \frac{1}{n}\delta_{\alpha\gamma} + \frac{1}{n^2}\left(-\frac{1-\beta}{2-\beta} + \frac{(1+\varepsilon_\alpha)(1+\varepsilon_\gamma)}{(1-\beta)(2-\beta)}\right) \quad (73)$$

The method above can be generalized to FP equations whose diffusion term is $\Gamma_{\alpha\gamma}\partial_\alpha\partial_\gamma P$ (as, for instance, in the low-temperature phase of our Ising model). The final result takes the form (72) with $(M_y)_{\alpha\gamma} = 2\Gamma_{\alpha'\gamma'}b_{\alpha\alpha'}b_{\gamma\gamma'}/(\lambda_\alpha + \lambda_\gamma)$.

Appendix B: entropy current distribution function for n heat baths

As in appendix A, we consider an Ising system split in n parts equilibrated with baths at inverse temperature β_α with $1 \leq \alpha \leq n$. The Lebowitz and Spohn [4] integrated entropy current (10) cannot be decomposed as in (11). We will thus study the distribution of $\mathcal{Q}_S(t)$. As in section 4 the corresponding function $\mu(\lambda)$ is given by the largest eigenvalue of some operator $-\hat{H}(\lambda)$, which reads :

$$\hat{H}(\lambda) = \sum_{\alpha=1}^n \sum_{j \in \alpha} \left(e^{-\sigma_j^z \beta_\alpha M_j^z / N} - \sigma_j^x e^{(2\lambda-1)\beta_\alpha \sigma_j^z M_j^z / N} \right) e^{-\beta_\alpha / N} \quad (74)$$

where we defined the operator $M_j^z = \sum_{i \neq j} \sigma_i^z$ verifying $[M_j^z, \sigma_j^x] = 0$. As expected on general grounds [3, 4], this operator possesses the Gallavotti-Cohen symmetry : for λ real, $(\hat{H}(\lambda))^\dagger = \hat{H}(1-\lambda)$. There is no need here to perform a symmetrization analogous to (44) because we directly focus on the *total* entropy current $\mathcal{Q}_S(t)$.

For the sake of simplicity, we will now confine our analysis to the disordered phase ($\beta < 1$). In that phase the magnetizations can be expanded up to order $\mathcal{O}(N^{-1/2})$ in terms of bosonic operators $a_\alpha, a_\alpha^\dagger$:

$$M_\alpha^x = N/n - 2a_\alpha^\dagger a_\alpha, \quad M_\alpha^y = -i\sqrt{N/n}(a_\alpha^\dagger - a_\alpha), \quad M_\alpha^z = \sqrt{N/n}(a_\alpha^\dagger + a_\alpha), \quad (75)$$

yielding the following expression, still bearing the Gallavotti-Cohen symmetry :

$$\begin{aligned} \hat{H}(\lambda) = & \frac{2}{n} \lambda(1-\lambda)(\beta^2 + \sigma^2) \left(\sum_{\alpha} (a_\alpha^\dagger + a_\alpha) \right)^2 \\ & - \frac{2}{n} \sum_{\alpha, \gamma} \beta_\gamma ((1-\lambda)a_\gamma^\dagger + \lambda a_\gamma)(a_\alpha^\dagger + a_\alpha) + 2 \sum_{\alpha} a_\alpha^\dagger a_\alpha + 2\lambda\beta \end{aligned} \quad (76)$$

where we define and recall that :

$$\sigma^2 = \frac{1}{n} \sum_{\alpha} \varepsilon_\alpha^2, \quad \beta = \frac{1}{n} \sum_{\alpha} \beta_\alpha \quad (77)$$

As in section 4, we find $\mu(\lambda)$ is given by :

$$\mu(\lambda) = \frac{1}{2} \int \frac{d\omega}{2\pi} \ln \frac{\det(\tilde{\Gamma} + \Omega)}{\det(\Gamma + \Omega)} - 2\lambda(1-\lambda)(\beta^2 + \sigma^2) \quad (78)$$

where $\Gamma, \tilde{\Gamma}$ and Ω are $2n \times 2n$ matrices defined by blocks in the following way :

$$\Gamma(\lambda) = 2 \begin{pmatrix} A & C^\dagger \\ C & A' \end{pmatrix}, \quad \tilde{\Gamma} = 2 \begin{pmatrix} 0 & C^\dagger \\ C & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & i\omega \text{Id} \\ -i\omega \text{Id} & 0 \end{pmatrix} \quad (79)$$

with elements :

$$A_{\alpha\gamma} = \frac{2}{n} \lambda(1-\lambda)(\beta^2 + \sigma^2) - \frac{\lambda}{n} (\beta_\alpha + \beta_\gamma) \quad (80)$$

$$A'_{\alpha\gamma} = \frac{2}{n} \lambda(1-\lambda)(\beta^2 + \sigma^2) - \frac{1-\lambda}{n} (\beta_\alpha + \beta_\gamma) \quad (81)$$

$$C_{\alpha\gamma} = \frac{2}{n} \lambda(1-\lambda)(\beta^2 + \sigma^2) - \frac{1}{n} ((1-\lambda)\beta_\alpha + \lambda\beta_\gamma) + \delta_{\alpha\gamma} \quad (82)$$

Tedious yet straightforward calculations lead to the following expressions :

$$\det(\Gamma + \Omega) = (-1)^n (\omega^2 + 4)^{n-2} (\omega^2 + \omega_+^2) (\omega^2 + \omega_-^2) \quad (83)$$

$$\det(\tilde{\Gamma} + \Omega) = (-1)^n (\omega^2 + 4)^{n-2} (\omega^2 + \tilde{\omega}_+^2) (\omega^2 + \tilde{\omega}_-^2) \quad (84)$$

$$\omega_{\pm} = \sqrt{(2 - \beta)^2 + 4\lambda(1 - \lambda)\sigma^2} \pm \sqrt{\beta^2 + 4\lambda(1 - \lambda)\sigma^2} \quad (85)$$

$$\begin{aligned} \tilde{\omega}_{\pm} &= 2 - \beta + 2\lambda(1 - \lambda)(\beta^2 + \sigma^2) \pm \\ &\quad \sqrt{4\lambda(1 - \lambda)\sigma^2 + (2\lambda(1 - \lambda)(\beta^2 + \sigma^2) - \beta)^2} \end{aligned} \quad (86)$$

The expression of $\mu(\lambda)$ simplifies into:

$$\mu(\lambda) = 2 - \beta - \sqrt{(2 - \beta)^2 + 4\lambda(1 - \lambda)\sigma^2} \quad (87)$$

We again obtain the full spectrum of the evolution operator:

$$\text{Sp } \hat{H}(\lambda) = \{2k + \omega_+(\lambda)\ell + \omega_-(\lambda)\ell' - \mu(\lambda)\}_{k,\ell,\ell' \in \mathbb{N}} \quad (88)$$

whose degeneracy is $\binom{k+n-3}{k}$. In the high temperature phase, the modes of the evolution operator split into two groups: two modes are similar to the $n = 2$ bath case, and $n - 2$ modes relax with constant rate 2. The slowest relaxation time of the master equation evolution operator is the same as that found for $n = 2$. The components of the probability current along the corresponding $n - 2$ directions cancel in the NESS, thus leaving the probability flow lines as ellipses, as is the case for $n = 2$.

Finally, we can remark that the result (87) for $\mu(\lambda)$ could also have been found by studying the set of Langevin equations associated to the Fokker–Planck equation of the system. However, the Langevin formalism does not render the Gallavotti–Cohen symmetry explicit at intermediate steps of the calculations, as opposed to the operator approach chosen in Sec. 4.

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